

ON THE NONVANISHING HYPOTHESIS FOR RANKIN-SELBERG CONVOLUTIONS FOR $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$

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ABSTRACT. Inspired by Sun's breakthrough in establishing the nonvanishing hypothesis for Rankin-Selberg convolutions for the groups $\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_{n-1}(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_{n-1}(\mathbb{C})$, we confirm it for $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ at the central critical point.

1. INTRODUCTION

The nonvanishing hypothesis is vital to the arithmetic study of critical values of higher degree L-functions and to the constructions of higher degree p-adic L-functions. Recently, Sun made a breakthrough by confirming it for $\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_{n-1}(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_{n-1}(\mathbb{C})$, see [16]. The current paper aims to consider the $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ case, which has been expected by Grenié since 2003, see page 284 of [5].

Fix an integer $n \geq 2$. Let \mathbb{K} be a topological field which is isomorphic to \mathbb{C} , and write

$$\iota_1, \iota_2 : \mathbb{K} \rightarrow \mathbb{C}$$

for two distinct isomorphisms.

Let $B_n(\mathbb{C}) = T_n(\mathbb{C})U_n(\mathbb{C})$ be the group of upper triangular matrices in $\mathrm{GL}_n(\mathbb{C})$. Here $T_n(\mathbb{C})$ is the standard maximal torus in $\mathrm{GL}_n(\mathbb{C})$, while $U_n(\mathbb{C})$ is the standard unipotent radical of $B_n(\mathbb{C})$. We identify \mathbb{Z}^n with the set of algebraic characters of $T_n(\mathbb{C})$ by sending $\mu = (\mu_i)$ to $t \mapsto \prod_i t_i^{\mu_i}$.

Fix a sequence of integers

$$\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n; \mu_{n+1} \geq \mu_{n+2} \geq \cdots \geq \mu_{2n}).$$

Denote by F_μ the irreducible algebraic representation of $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ with highest weight μ . It is also viewed as an irreducible representation of the real Lie group $\mathrm{GL}_n(\mathbb{K})$ via the complexification map

$$(1) \quad \mathrm{GL}_n(\mathbb{K}) \rightarrow \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}), \quad g \mapsto (\iota_1(g), \iota_2(g)).$$

As usual, we do not distinguish a representation with its underlying space.

Recall that a representation of a real reductive group is called a *Casselman-Wallach representation* if it is smooth, Fréchet, of moderate growth, and its Harish-Chandra module has finite length, see [2] and Chapter 11 of [19] for more details. Denote by $\Omega(\mu)$ the set of isomorphism classes of irreducible Casselman-Wallach representations π of $\mathrm{GL}_n(\mathbb{K})$ such that

- $\pi|_{\mathrm{SL}_n(\mathbb{K})}$ is unitarizable and tempered; and

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- the relative Lie algebra cohomology

$$(2) \quad H^\bullet(\mathfrak{gl}_n(\mathbb{C}) \oplus \mathfrak{gl}_n(\mathbb{C}), \mathrm{GU}(n); \pi \otimes F_\mu^\vee) \neq 0,$$

where $\mathfrak{gl}_n(\mathbb{C}) \oplus \mathfrak{gl}_n(\mathbb{C})$ is viewed as the complexification of $\mathfrak{gl}_n(\mathbb{K})$ through the differential of (1), and

$$\mathrm{GU}(n) := \{g \in \mathrm{GL}_n(\mathbb{K}) \mid \iota_1(g)\iota_2(g)^\dagger \text{ is a scalar matrix}\}.$$

According to Section 3 of [3], we have

$$(3) \quad \#\Omega(\mu) = \begin{cases} 0, & \text{if } \mu \text{ is not pure;} \\ 1, & \text{if } \mu \text{ is pure.} \end{cases}$$

Here “ μ is pure” means that there is an integer w such that

$$(4) \quad \mu_1 + \mu_{2n} = \mu_2 + \mu_{2n-1} = \cdots = \mu_n + \mu_{n+1} = w.$$

In such a case, we shall say that μ is *pure with weight* w . Assume that μ is pure, and let π_μ be the unique representation in $\Omega(\mu)$.

Put

$$(5) \quad b_n := \frac{n(n-1)}{2}.$$

Then by Lemma 3.14 of [3], we have

$$H^b(\mathfrak{gl}_n(\mathbb{C}) \oplus \mathfrak{gl}_n(\mathbb{C}), \mathrm{GU}(n); \pi_\mu \otimes F_\mu^\vee) = 0, \quad \text{if } b < b_n,$$

and

$$\dim H^{b_n}(\mathfrak{gl}_n(\mathbb{C}) \oplus \mathfrak{gl}_n(\mathbb{C}), \mathrm{GU}(n); \pi_\mu \otimes F_\mu^\vee) = 1.$$

We fix another sequence of integers

$$\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n; \nu_{n+1} \geq \nu_{n+2} \geq \cdots \geq \nu_{2n}).$$

Assume that ν is pure with weight w' . Define F_ν , $\Omega(\nu)$ and π_ν similarly.

Write

$$G := \mathrm{GL}_n(\mathbb{K}) \quad \text{and} \quad \tilde{K} := \mathrm{GU}(n).$$

Let P be the standard maximal parabolic subgroup of G of type $(n-1, 1)$. Let “ $|\cdot|_{\mathbb{K}}$ ” denotes the normalized absolute value of \mathbb{K} . That is, $|z|_{\mathbb{K}} = \iota_1(z)\iota_2(z)$ for $z \in \mathbb{K}$. Let

$$H_s(p) := \delta_P^s(p) \cdot \eta(p_{nn})^{-1} = |\det p|_{\mathbb{K}}^s \cdot |p_{nn}|_{\mathbb{K}}^{-ns} \cdot \eta(p_{nn})^{-1},$$

where $\eta = \omega_{\pi_\mu} \omega_{\pi_\nu}$ is the product of the central characters of π_μ and π_ν . For $s \in \mathbb{C}$, define the normalized smooth induced representation

$$(6) \quad I_s := \mathrm{Ind}_P^G \left(H_{s-\frac{1}{2}} \right).$$

Write

$$G^3 := G \times G \times G \quad \text{and} \quad \tilde{K}^3 := \tilde{K} \times \tilde{K} \times \tilde{K},$$

then G (resp. \tilde{K}) embeds in G^3 (resp. \tilde{K}^3) diagonally. Here and henceforth, we use the corresponding lower case gothic letter to indicate the complexified Lie algebra of a Lie group.

Assume that μ and ν are compatible (see Definition 3.6), and that $j \in \mathbb{Z}$ is a critical place (see Section 2) for $\pi_\mu \times \pi_\nu$. Denote

$$\pi_\xi := \pi_\mu \hat{\otimes} \pi_\nu \hat{\otimes} I_j \quad (\text{the completed projective space product}).$$

By Section 2 of [6], the Rankin-Selberg integrals produce a nonzero element

$$(7) \quad \phi_\pi \in \mathrm{Hom}_G(\pi_\xi, \mathbb{C}).$$

Let V_j be the finite dimensional representation of $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ defined in Proposition 3.3 (b) or (c), and put

$$(8) \quad c_n = n - 1.$$

Then in the setting of Proposition 3.2 (b) and (c), we have

$$\mathrm{H}^c(\mathfrak{g}, \tilde{K}; I_j \otimes V_j) = 0, \quad \text{if } c < c_n,$$

and

$$\dim \mathrm{H}^{c_n}(\mathfrak{g}, \tilde{K}; I_j \otimes V_j) = 1.$$

Denote

$$(9) \quad F_\xi^\vee := F_\mu^\vee \otimes F_\nu^\vee \otimes V_j.$$

Then there is a nonzero element

$$(10) \quad \phi_F \in \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})}(F_\xi^\vee, \mathbb{C}).$$

By Künneth formula,

$$\dim \mathrm{H}^{2b_n+c_n}(\mathfrak{g}^3, \tilde{K}^3; \pi_\xi \otimes F_\xi^\vee) = 1.$$

Note that

$$\dim_{\mathbb{C}}(\mathfrak{g}/\tilde{\mathfrak{k}}) = n^2 - 1 = 2b_n + c_n.$$

It follows that

$$\dim \mathrm{H}^{2b_n+c_n}(\mathfrak{g}, \tilde{K}; \mathbb{C}) = 1.$$

Put

$$(11) \quad \kappa := \frac{w + w'}{2}.$$

Our main result is as follows, which can be viewed as the nonvanishing hypothesis for $\mathrm{GL}_n \times \mathrm{GL}_n$ at the critical place $j = -\kappa + \frac{1}{2}$.

Theorem A. *Assume that*

$$(12) \quad \kappa \text{ is an half integer and fix } j = -\kappa + \frac{1}{2}.$$

By restriction of cohomology, the G -equivariant linear functional

$$(13) \quad \phi_\pi \otimes \phi_F : \pi_\xi \otimes F_\xi^\vee \rightarrow \mathbb{C} = \mathbb{C} \otimes \mathbb{C}$$

induces a nonzero map

$$(14) \quad \mathrm{H}^{2b_n+c_n}(\mathfrak{g}^3, \tilde{K}^3; \pi_\xi \otimes F_\xi^\vee) \rightarrow \mathrm{H}^{2b_n+c_n}(\mathfrak{g}, \tilde{K}; \mathbb{C}).$$

Put $j_0 = -\kappa + \frac{1}{2}$. We remark that I_{j_0} is unitary when restricted to $SL(n, \mathbb{K})$. For a general critical place j of $\pi_\mu \times \pi_\nu$, we succeeded in using the translation functor (see Chapter 7 of [10]) to realize I_j as a submodule of $I_{j_0} \otimes F_j$ with multiplicity one, where F_j is certain finite dimensional representation of \mathfrak{g} . However, when $j \neq j_0$, the lowest K -type of I_j has

multiplicity greater than one in $I_{j_0} \otimes F_j$. This prevents us from obtaining an analogue of Proposition 3.5 of [16].

The outline of the paper is as follows: We recall the definition of critical places and calculated them for $\pi_\mu \times \pi_\nu$ in Section 2. Then we recall some known results and describe compatibility in a clean fashion in Section 3. We collect all the necessary analysis of finite dimensional representations in Section 4, while those for infinite dimensional representations are presented in Section 5. Then after a short discussion of relative Lie algebra cohomology spaces in Section 6, we prove Theorem A in Section 7.

Finally, we remark that although we have followed Sun's pioneering paper [16] closely for the general approach, there are still many delicate analysis to carry out in our case.

2. THE CRITICAL PLACES

Assume that μ is pure with weight w , see (4). We write $\mu = (\mu^L; \mu^R)$, where

$$(15) \quad \mu^L = (\mu_1, \dots, \mu_n), \quad \mu^R = (\mu_{n+1}, \dots, \mu_{2n}) = (w - \mu_n, \dots, w - \mu_1).$$

As in Section 2.4 of [13], we put

$$(16) \quad a_i = \mu_i + \frac{n+1-2i}{2}, \quad b_i = w - a_i, \quad 1 \leq i \leq n.$$

Now define the representation J_μ to be induced from the Bore subgroup $B_n(\mathbb{C})$ of upper triangular matrices as:

$$(17) \quad J_\mu := \text{Ind}_{B_n(\mathbb{C})}^{\text{GL}_n(\mathbb{C})} \left(z_{11}^{a_1} \bar{z}_{11}^{b_1} \otimes \dots \otimes z_{nn}^{a_n} \bar{z}_{nn}^{b_n} \right),$$

where for any half-integers a, b , $z^a \bar{z}^b$ stands for the character of \mathbb{C}^\times sending z to $z^a \bar{z}^b$. It is known that J_μ is the unique representation in $\Omega(\mu)$, see Proposition 2.14 of [13]. Thus we can take π_μ to be J_μ .

Similarly, assume that ν is pure with weight w' and write $\nu = (\nu^L; \nu^R)$, where

$$(18) \quad \nu^L = (\nu_1, \dots, \nu_n), \quad \nu^R = (\nu_{n+1}, \dots, \nu_{2n}) = (w' - \nu_n, \dots, w' - \nu_1).$$

Put

$$(19) \quad c_i = \nu_i + \frac{n+1-2i}{2}, \quad d_i = w' - c_i, \quad 1 \leq i \leq n.$$

Now we can take the representation π_ν to be the following representation:

$$(20) \quad J_\nu := \text{Ind}_{B_n(\mathbb{C})}^{\text{GL}_n(\mathbb{C})} \left(z_{11}^{c_1} \bar{z}_{11}^{d_1} \otimes \dots \otimes z_{nn}^{c_n} \bar{z}_{nn}^{d_n} \right).$$

Definition 2.1. An integer s_0 is called a *critical place* for $\pi_\mu \times \pi_\nu$ if neither $L(\pi_\mu \times \pi_\nu; s)$ nor $L(\pi_\mu^\vee \times \pi_\nu^\vee; 1-s)$ has a pole at s_0 .

We are going to determine the set of critical places for $\pi_\mu \times \pi_\nu$. Define

$$(21) \quad c_{\mu, \nu} = \min_{1 \leq i, j \leq n} |\mu_i + \nu_j - \kappa + (n+1) - (i+j)|.$$

Proposition 2.2. *The integer s is a critical place for $\pi_\mu \times \pi_\nu$ if and only if*

$$(22) \quad 1 - \kappa - c_{\mu, \nu} \leq s \leq -\kappa + c_{\mu, \nu}.$$

Proof. By (16), (19) and Section 4 of [8], we have

$$\begin{aligned} L(\pi_\mu \otimes \pi_\nu; s) &\sim \prod_{i=1}^n \prod_{j=1}^n \Gamma\left(s + \frac{a_i + b_i + c_i + d_i}{2} + \frac{|a_i - b_i + c_i - d_i|}{2}\right) \\ &= \prod_{i=1}^n \prod_{j=1}^n \Gamma\left(s + \kappa + \frac{|a_i - b_i + c_i - d_i|}{2}\right) \\ &= \prod_{i=1}^n \prod_{j=1}^n \Gamma(s + \kappa + |\mu_i + \nu_j - \kappa + (n+1) - (i+j)|), \end{aligned}$$

where “ \sim ” means equality up to a nonzero real number. Similarly, we have

$$L(\pi_\mu^\vee \otimes \pi_\nu^\vee; 1-s) \sim \prod_{i=1}^n \prod_{j=1}^n \Gamma(1-s - \kappa + |\mu_i + \nu_j - \kappa + (n+1) - (i+j)|).$$

Since the gamma function has simple poles at non-positive integers, we have that the integer s is a critical place for $\pi_\mu \times \pi_\nu$ if and only if

$$s + \kappa + |\mu_i + \nu_j - \kappa + (n+1) - (i+j)| \geq 1,$$

and

$$1 - s - \kappa + |\mu_i + \nu_j - \kappa + (n+1) - (i+j)| \geq 1$$

for all $1 \leq i, j \leq n$. Then one arrives at (22) immediately. \square

3. COMPATIBILITY

3.1. Let $G = GL(n, \mathbb{K})$. Then $G_{\mathbb{C}} = GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$, $\mathfrak{g}_0 = \mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{gl}(n, \mathbb{C})$. Take T be the the Cartan subgroup consisting of the diagonal matrices. Let P be the standard maximal parabolic subgroup of type $(n-1, 1)$. Then the Levi factor L of P has the form $A \times M$, where A consists of diagonal matrices with diagonal entries a_1, \dots, a_1, a_2 ; while M consists of block-diagonal matrices with size $(n-1, 1)$. We fix a set of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{t})$ for $\Delta(\mathfrak{g}, \mathfrak{t})$, and choose compatible positive root systems for P and L . Let $W = W(\mathfrak{g}, \mathfrak{t})$ be the Weyl group of \mathfrak{g} with respect to \mathfrak{t} , and similarly $W_L = W(\mathfrak{l}, \mathfrak{t})$.

Recall that for $j \in \mathbb{Z}$, we have the representation $I_j = \text{Ind}_P^G(H_{j-\frac{1}{2}})$, where $H_s(p) = |\det p|_{\mathbb{K}}^s \cdot |p_{nn}|_{\mathbb{K}}^{-ns} \cdot \eta(p_{nn})^{-1}$. Here $\eta(z) = \omega_{\pi_\mu}(z) \omega_{\pi_\nu}(z)$ is the product of the central characters of π_μ and π_ν . Put

$$(23) \quad k_\eta := \sum_{i=1}^n (a_i + c_i) = \sum_{i=1}^n (\mu_i + \nu_i).$$

By (17) and (20), we have that

$$\eta(z) = z^{\sum_{i=1}^n (a_i + c_i)} \bar{z}^{\sum_{i=1}^n (b_i + d_i)} = z^{k_\eta} \bar{z}^{2n\kappa - k_\eta}.$$

For any $k \in \mathbb{Z}$, we denote by \det^k the representation of $GL(n, \mathbb{C})$ with highest weight (k, k, \dots, k) . For $a \in \mathbb{N}$, we denote by Sym^a the representation of $GL(n, \mathbb{C})$ with highest weight $(a, 0, \dots, 0)$; while if $a < 0$, we denote by Sym^a the representation of $GL(n, \mathbb{C})$ with highest weight $(0, 0, \dots, a)$.

Let us record Proposition 2 and Corollary 1 of [5] for later usage.

Proposition 3.1. (Grenié) *Let σ and τ be two finite dimensional irreducible representations of $GL(n, \mathbb{C})$ with highest weights $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_n)$, respectively. Then \det^d occurs in $\tau \otimes \sigma \otimes \text{Sym}^a$ ($a \in \mathbb{N}$) if and only if $\max_{1 \leq i \leq n} \{\lambda_i + \mu_{n+1-i}\} \leq d \leq \min_{1 \leq i \leq n-1} \{\lambda_i + \mu_{n-i}\}$ and $a = nd - \sum_{i=1}^n (\mu_i + \lambda_i) \geq 0$. Moreover, in such a case, \det^d occurs with multiplicity one in $\tau \otimes \sigma \otimes \text{Sym}^a$.*

The above proposition is deduced from the Pieri's rule, see Corollary 9.2.4 of [4]. The following result can be deduced as Proposition 5 of [5], where the main tool is Theorem III.3.3 of [1].

Proposition 3.2. (Grenié) *The relative Lie algebra cohomology $H^*(\mathfrak{g}, K; I_j \otimes V_j)$ is non-vanishing if and only if one of the following happens:*

$$(a) \quad j \geq \max \left\{ 1 - \frac{k_\eta}{n}, 1 + \frac{k_\eta}{n} - 2\kappa \right\}, \text{ and}$$

$$V_j = (\text{Sym}^{nj+k_\eta-n} \otimes \det^{1-j}; \text{Sym}^{nj-n-k_\eta+2n\kappa} \otimes \det^{1-j}).$$

We then take $l(j) = 2(n-1)$.

$$(b) \quad 1 - 2\kappa + \frac{k_\eta}{n} \leq j \leq -\frac{k_\eta}{n}, \text{ and}$$

$$V_j = (\det^{-j} \otimes \text{Sym}^{nj+k_\eta}; \text{Sym}^{nj-n-k_\eta+2n\kappa} \otimes \det^{1-j}).$$

We then take $l(j) = n-1$.

$$(c) \quad 1 - \frac{k_\eta}{n} \leq j \leq -2\kappa + \frac{k_\eta}{n}, \text{ and}$$

$$V_j = (\text{Sym}^{nj+k_\eta-n} \otimes \det^{1-j}; \det^{-j} \otimes \text{Sym}^{nj+2n\kappa-k_\eta}).$$

We then take $l(j) = n-1$.

$$(d) \quad j \leq \min \left\{ -\frac{k_\eta}{n}, \frac{k_\eta}{n} - 2\kappa \right\}, \text{ and}$$

$$V_j = (\det^{-j} \otimes \text{Sym}^{nj+k_\eta}; \det^{-j} \otimes \text{Sym}^{nj+2n\kappa-k_\eta}).$$

We then take $l(j) = 0$.

The cohomology is equal to \mathbb{C} in degree $l(j)$, \mathbb{C}^2 in degree $l(j) + 1$, \mathbb{C} in degree $l(j) + 2$ and zero in all other degrees.

We put

$$(24) \quad M_{\mu, \nu}^n := \max_{\substack{i+j=n \\ 1 \leq i, j \leq n}} \{\mu_i + \nu_j\}, \quad m_{\mu, \nu}^n := \min_{\substack{i+j=n \\ 1 \leq i, j \leq n}} \{\mu_i + \nu_j\}.$$

The numbers $M_{\mu, \nu}^{n+1}$, $m_{\mu, \nu}^{n+1}$, $M_{\mu, \nu}^{n+2}$, $m_{\mu, \nu}^{n+2}$ are interpreted similarly.

Proposition 3.3. (i) *In the setting of Proposition 3.2(b), we have*

$$\dim \text{Hom}_{G_{\mathbb{C}}}(F_{\mu}^{\vee} \otimes F_{\nu}^{\vee} \otimes V_j, \mathbb{C}) \leq 1.$$

Moreover, equality holds if and only if

$$(25) \quad -m_{\mu, \nu}^n \leq j \leq -M_{\mu, \nu}^{n+1} \quad \text{and} \quad 1 - 2\kappa + M_{\mu, \nu}^{n+1} \leq j \leq 1 - 2\kappa + m_{\mu, \nu}^n.$$

(ii) *In the setting of Proposition 3.2(c), we have*

$$\dim \text{Hom}_{G_{\mathbb{C}}}(F_{\mu}^{\vee} \otimes F_{\nu}^{\vee} \otimes V_j, \mathbb{C}) \leq 1.$$

Moreover, equality holds if and only if

$$(26) \quad 1 - m_{\mu,\nu}^{n+1} \leq j \leq 1 - M_{\mu,\nu}^{n+2} \quad \text{and} \quad M_{\mu,\nu}^{n+2} - 2\kappa \leq j \leq -2\kappa + m_{\mu,\nu}^{n+1}.$$

Proof. For (i), note that

$$\begin{aligned} \text{Hom}_{G_{\mathbb{C}}}(F_{\mu}^{\vee} \otimes F_{\nu}^{\vee} \otimes V_j, \mathbb{C}) &= \text{Hom}_{G_{\mathbb{C}}}(V_j, F_{\mu} \otimes F_{\nu}) \\ &= \text{Hom}_G(\det^{-j} \otimes \text{Sym}^{nj+k_{\eta}}, E_{\mu L} \otimes E_{\nu L}) \\ &\times \text{Hom}_G(\text{Sym}^{nj-n+2n\kappa-k_{\eta}} \otimes \det^{1-j}, E_{\mu R} \otimes E_{\nu R}) \\ &= \text{Hom}_G(\det^{-j}, E_{\mu L} \otimes E_{\nu L} \otimes \text{Sym}^{-nj-k_{\eta}}) \\ &\times \text{Hom}_G(E_{\mu R}^{\vee} \otimes E_{\nu R}^{\vee} \otimes \text{Sym}^{nj-n+2n\kappa-k_{\eta}}, \det^{j-1}). \end{aligned}$$

Here E_{λ} denotes the representation of $GL(n, \mathbb{C})$ with highest weight λ . By Proposition 3.1, one has that $\text{Hom}_G(\det^{-j}, E_{\mu L} \otimes E_{\nu L} \otimes \text{Sym}^{-nj-k_{\eta}})$ is non-vanishing if and only if

$$\max_{1 \leq i \leq n} \{\mu_i + \nu_{n+1-i}\} \leq -j \leq \min_{1 \leq i \leq n-1} \{\mu_i + \nu_{n-i}\};$$

and that $\text{Hom}_G(E_{\mu R}^{\vee} \otimes E_{\nu R}^{\vee} \otimes \text{Sym}^{nj-n+2n\kappa-k_{\eta}}, \det^{j-1})$ is non-vanishing if and only if

$$\max_{1 \leq i \leq n} \{\mu_i + \nu_{n+1-i}\} - 2\kappa \leq j - 1 \leq \min_{1 \leq i \leq n-1} \{\mu_i + \nu_{n-i}\} - 2\kappa.$$

Thus one arrives at (25), as desired.

The proof for (ii) is similar, we omit it here. \square

Lemma 3.4. (i) Suppose that (25) has solutions, then they coincide with those to (22).
(ii) Suppose that (26) has solutions, then they coincide with those to (22).

Proof. It is easy to see that (25) has solutions if and only if

$$M_{\mu,\nu}^{n+1} \leq m_{\mu,\nu}^n, \quad 1 - 2\kappa + M_{\mu,\nu}^{n+1} \leq -M_{\mu,\nu}^{n+1}, \quad -m_{\mu,\nu}^n \leq 1 - 2\kappa + m_{\mu,\nu}^n,$$

if and only if

$$(27) \quad M_{\mu,\nu}^{n+1} \leq \kappa - \frac{1}{2} \leq m_{\mu,\nu}^n.$$

Now suppose that (27) holds. On one hand, for any i, j such that $i + j \leq n$, we have

$$\mu_i + \nu_j - \kappa + (n + 1) - (i + j) \geq m_{\mu,\nu}^n - \kappa + 1 \geq \frac{1}{2} > 0.$$

On the other hand, for any i, j such that $i + j \geq n + 1$, we have

$$\mu_i + \nu_j - \kappa + (n + 1) - (i + j) \leq M_{\mu,\nu}^{n+1} - \kappa \leq -\frac{1}{2} < 0.$$

We conclude that

$$c_{\mu,\nu} = \min \{m_{\mu,\nu}^n - \kappa + 1, \kappa - M_{\mu,\nu}^{n+1}\}.$$

Then one sees that (22) is just a reformulation of (25). This proves (i).

For (ii), it is easy to see that (26) has solutions if and only if

$$(28) \quad M_{\mu,\nu}^{n+2} \leq \kappa + \frac{1}{2} \leq m_{\mu,\nu}^{n+1}.$$

The remaining discussion is similar to the previous case, we omit the details. \square

Remark 3.5. Note that if (27) holds, we have

$$\frac{k_\eta}{n} = \frac{1}{n} \sum_{i=1}^n (\mu_i + \nu_{n+1-i}) \leq M_{\mu,\nu}^{n+1} \leq \kappa - \frac{1}{2}.$$

Then one sees that “ $1 - 2\kappa + \frac{k_\eta}{n} \leq j \leq -\frac{k_\eta}{n}$ ” holds since (27) is equivalent to (25). Similarly, if (28) holds, we have

$$\frac{k_\eta}{n} = \frac{1}{n} \sum_{i=1}^n (\mu_i + \nu_{n+1-i}) \geq m_{\mu,\nu}^{n+1} \geq \kappa + \frac{1}{2}.$$

Then one sees that “ $1 - \frac{k_\eta}{n} \leq j \leq -2\kappa + \frac{k_\eta}{n}$ ” holds since (28) is equivalent to (26).

Definition 3.6. We say that two pure weights μ and ν are *compatible* if (25) or (26) has solutions.

By Lemma 3.4 and Remark 3.5, we see that if μ and ν are compatible, the solutions to (25) or (26) coincide with the critical places for $\pi_\mu \times \pi_\nu$.

4. FINITE DIMENSIONAL REPRESENTATIONS

4.1. Cartan component and PRV component. In this subsection, we let \mathfrak{g} be a finite dimensional simple Lie algebra over \mathbb{C} . Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} . Fix a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{h})$. Then for any dominant integral weight $\lambda \in \mathfrak{h}^*$, we denote by F_λ the finite dimensional irreducible representation of \mathfrak{g} with highest weight λ . Let $W = W(\mathfrak{g}, \mathfrak{h})$ be the Weyl group. Take w_0 as the longest element of W . Then F_λ has lowest weight $w_0\lambda$. Let F_μ be another irreducible representation of \mathfrak{g} with highest weight μ . Then it is well-known that $F_{\lambda+\mu}$ occurs with multiplicity one in $F_\lambda \otimes F_\mu$. We call $F_{\lambda+\mu}$ the *Cartan component* of $F_\lambda \otimes F_\mu$. On the other hand, let σ be the irreducible representation of \mathfrak{g} with extremal weight $\lambda + w_0\mu$. Then by Corollary 1 to Theorem 2 of [12], we have that σ occurs with multiplicity one in $F_\lambda \otimes F_\mu$. We call σ the *PRV component* of $F_\lambda \otimes F_\mu$. Here, “PRV” stands for the initials of Parthasarathy, Rao, and Varadarajan.

4.2. Minimal K -types. Recall that \mathbb{K} is a topological field isomorphic to \mathbb{C} . We put $G = GL(n, \mathbb{K})$, $K = U(n)$ and $\tilde{K} = GU(n)$. Recall from (1) that $G_{\mathbb{C}} \cong GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$, and that $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}) \times \mathfrak{gl}(n, \mathbb{C})$ is the complexified Lie algebra of G . Fix a Borel subgroup B of G consisting of the upper triangular matrices, and choose a positive root system for G accordingly. We use E_λ to denote the irreducible G representation with highest weight λ . Weyl’s unitary trick (see Proposition 7.15 of [9]) allows us to shift freely between $GL(n, \mathbb{K})$ representations and $U(n)$ representations. Thus we shall abuse notation slightly, and denote by E_λ the K -type with highest weight λ as well.

From now on, we always assume that μ and ν are compatible, and that $j \in \mathbb{Z}$ is a critical place for $\pi_\mu \times \pi_\nu$. Let us revisit the representation I_j of G , which is defined in (6). The following lemma determines the K -types of I_j .

Lemma 4.1. *The K -types in I_j are exactly those with highest weights $(m, 0, \dots, 0, 2n\kappa - 2k_\eta - m)$, where the integer $m \geq \max\{0, 2n\kappa - 2k_\eta\}$. Moreover, $I_j|_K$ is multiplicity free.*

Proof. Recall that $H_j(p) = |\det p|_{\mathbb{K}}^j \cdot |p_{nn}|_{\mathbb{K}}^{-nj} \cdot \eta(p_{nn})^{-1}$, where $\eta(z)^{-1} = z^{-k_\eta} \bar{z}^{k_\eta - 2n\kappa}$. Let V be any K -type. By Frobenius reciprocity, we have

$$(29) \quad \text{Hom}_K(I_j, V) \cong \text{Hom}_{P \cap K}(H_j|_{P \cap K}, V|_{P \cap K}).$$

Note that $P \cap K = U(n-1) \times U(1)$, and that $H_j|_{P \cap K}$ has weight $(0, \dots, 0, 2n\kappa - 2k_\eta)$. Suppose that V has highest weight $(\lambda_1, \dots, \lambda_n)$. Then $V|_{P \cap K}$ contains $H_j|_{P \cap K}$ if and only if

$$\lambda_1 \geq 0 \geq \lambda_2 \geq 0 \geq \dots \geq 0 \geq \lambda_{n-1} \geq 0 \geq \lambda_n,$$

and that

$$\sum_{i=1}^n \lambda_i = 2n\kappa - 2k_\eta.$$

Therefore,

$$\lambda_1 = m, \lambda_2 = \dots = \lambda_{n-1} = 0, \lambda_n = 2n\kappa - 2k_\eta - m,$$

where $m \geq \max\{0, 2n\kappa - 2k_\eta\}$. \square

Denote the minimal K -type of I_j by σ_j^+ . In case (b) of Proposition 3.2, we have $2n\kappa - 2k_\eta \geq n$. Thus by Lemma 4.1,

$$(30) \quad \sigma_j^+ = E_{(2n\kappa - 2k_\eta, 0, \dots, 0)}.$$

In case (c) of Proposition 3.2, we have $2n\kappa - 2k_\eta \leq -n$. Thus by Lemma 4.1,

$$(31) \quad \sigma_j^+ = E_{(0, \dots, 0, 2n\kappa - 2k_\eta)}.$$

Lemma 4.2. *The minimal K -type, denoted by τ_μ^+ , of J_μ is the one with highest weight $(2\mu_1 - w + n - 1, \dots, 2\mu_n - w - n + 1)$. Moreover, τ_μ^+ occurs with multiplicity one in J_μ .*

This lemma is also deduced from the Frobenius reciprocity, we omit the details.

4.3. Analysis of the multiplicity of certain $GL(n, \mathbb{K})$ representations. Let $\tilde{\mathfrak{p}}$ be the Lie algebra of traceless matrices $M \in \mathfrak{g}$ such that $\overline{M}^t = M$. Then we have that

$$\mathfrak{g}/\tilde{\mathfrak{k}} \cong \tilde{\mathfrak{p}}$$

as K representations.

Let V be the standard representation of G , and let V^\vee be its contragredient. Let e_1, \dots, e_n be the standard basis of V , and let e_1^*, \dots, e_n^* be the corresponding dual dual basis. Then it is easy to check that the vector $e_{ij} := e_i \otimes e_j^*$ has weight $\epsilon_i - \epsilon_j$. Here $\epsilon_i = (0, \dots, 1, \dots, 0)$, where the unique 1 is the i -th entry. Now as proved in Proposition 6 of [5],

$$(32) \quad V \otimes V^\vee \cong \mathbb{C} \oplus \tilde{\mathfrak{p}}$$

and as K representations,

$$(33) \quad \tilde{\mathfrak{p}} \cong E_{(1, 0, \dots, 0, -1)}.$$

Indeed, the vector $\sum_{i=1}^n e_{ii}$ generates the trivial representation \mathbb{C} . On the other hand, one can check that e_{1n} is a highest weight vector, and that the $\dim E_{(1, 0, \dots, 0, -1)} = n^2 - 1$. This verifies (32) as well as (33). Let us fix a basis consisting of weight vectors of $\tilde{\mathfrak{p}}$:

$$(34) \quad e_{ij}, 1 \leq i \neq j \leq n; e_{11} - e_{kk}, 2 \leq k \leq n.$$

Lemma 4.3. *The $GL(n, \mathbb{K})$ representation $E_{(n-1, -1, \dots, -1)}$ and its contragredient both occur with multiplicity one in $\wedge^{n-1}(\mathfrak{g}/\tilde{\mathfrak{k}})$.*

Proof. Firstly, it is direct to check that

$$(35) \quad e_{12} \wedge \cdots \wedge e_{1n} \in \wedge^{n-1} \tilde{\mathfrak{p}}$$

is a highest weight vector with weight $(n-1, -1, \dots, -1)$. Thus $E_{(n-1, -1, \dots, -1)}$ occurs in $\wedge^{n-1} \tilde{\mathfrak{p}} \cong \wedge^{n-1}(\mathfrak{g}/\tilde{\mathfrak{k}})$.

Now suppose that v_0 is the highest weight vector of an occurrence of $E_{(n-1, -1, \dots, -1)}$ in $\wedge^{n-1} \tilde{\mathfrak{p}}$. Take an arbitrary linear component of v_0 , we may and we will assume that it has the form $A_1 \wedge \cdots \wedge A_{n-1}$, where each A_i is from (34). Observe that there is no repetitions among these A_i , and all of them are weight vectors. We collect the corresponding *non-zero* weights as $\alpha_1, \dots, \alpha_r$, which must be pairwise distinct. We have $r \leq n-1$, and

$$\alpha_1 + \cdots + \alpha_r = (n-1, -1, \dots, -1).$$

One sees that this equation has solution if and only if $r = n-1$ and

$$\{\alpha_1, \dots, \alpha_{n-1}\} = \{\epsilon_1 - \epsilon_2, \dots, \epsilon_1 - \epsilon_n\}.$$

Therefore, $A_1 \wedge \cdots \wedge A_{n-1}$ is a scalar multiple of (35). Then so is v_0 . Thus $E_{(n-1, -1, \dots, -1)}$ occurs with multiplicity one in $\wedge^{n-1} \tilde{\mathfrak{p}}$.

Since $\tilde{\mathfrak{p}}$ is self-dual as a G representation, we conclude immediately that its contragredient representation $E_{(1, \dots, 1, 1-n)}$ occurs exactly once in $\wedge^{n-1} \tilde{\mathfrak{p}}$ as well. \square

Remark 4.4. It is not hard to prove that a vector of the following form occurs in the unique realization of $E_{(n-1, 1, \dots, 1)}$ in $\wedge^{n-1}(\mathfrak{g}/\tilde{\mathfrak{k}})$:

$$(36) \quad (e_{11} - e_{22}) \wedge \cdots \wedge (e_{11} - e_{nn}) + \text{other terms},$$

where each linear component in “other terms” is different from the leading one.

Recall that the $G_{\mathbb{C}}$ representation F_{μ} has highest weight (μ_L, μ_R) . Define the K -type

$$(37) \quad \tau_{\mu} := E_{\mu_L - w_0 \mu_R} = E_{(2\mu_1 - w, \dots, 2\mu_n - w)},$$

where w_0 is the longest element of S_n . We define τ_{ν} similarly. Recall that the minimal K -type of J_{μ} is τ_{μ}^+ . We define τ_n to be the PRV-component of $\tau_{\mu}^{\vee} \otimes \tau_{\mu}^+$. By Lemma 4.2 and (37), we have

$$(38) \quad \tau_n := E_{(n-1, n-3, \dots, 1-n)}.$$

In particular, it is independent of μ . Thus PRV-component of $\tau_{\nu}^{\vee} \otimes \tau_{\nu}^+$ is τ_n as well.

Lemma 4.5. *The K -type τ_n occurs with multiplicity one in $\wedge^{b_n}(\mathfrak{g}/\tilde{\mathfrak{k}})$.*

Proof. Recall that $b_n = \frac{n(n-1)}{2}$. Firstly, it is direct to check that

$$(39) \quad e_{12} \wedge \cdots \wedge e_{1n} \wedge e_{23} \wedge \cdots \wedge e_{2n} \wedge \cdots \wedge e_{n-1, n} \in \wedge^{b_n} \tilde{\mathfrak{p}}$$

is a highest weight vector with weight $(n-1, n-3, \dots, 3-n, 1-n)$. Thus τ_n occurs in $\wedge^{b_n} \tilde{\mathfrak{p}} \cong \wedge^{b_n}(\mathfrak{g}/\tilde{\mathfrak{k}})$.

Now suppose that v_0 is the highest weight vector of an occurrence of τ_n in $\wedge^{b_n} \tilde{\mathfrak{p}}$. Take an arbitrary linear component of v_0 , we may and we will assume that it has the form

$A_1 \wedge \cdots \wedge A_{b_n}$, where each A_i is from (34). Observe that there is no repetitions among those A_i , and all of them are weight vectors. We collect the corresponding *non-zero* weights as $\alpha_1, \dots, \alpha_r$, which must be pairwise distinct. We have $r \leq b_n$, and

$$\alpha_1 + \cdots + \alpha_r = (n-1, n-3, \dots, 3-n, 1-n).$$

Note that the RHS is the sum of all the positive roots. Thus this equation has solution if and only if $r = b_n$ and

$$\{\alpha_1, \dots, \alpha_{b_n}\} = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\}.$$

Therefore, $A_1 \wedge \cdots \wedge A_{b_n}$ is a scalar multiple of (39). Then so is v_0 . We conclude that τ_n occurs with multiplicity one in $\wedge^{b_n} \tilde{\mathfrak{p}}$. \square

Remark 4.6. One sees easily that the vector $e_{21} \wedge \cdots \wedge e_{n1} \wedge e_{32} \wedge \cdots \wedge e_{n2} \wedge \cdots \wedge e_{n,n-1}$ occurs in the unique realization of τ_n in $\wedge^{b_n}(\mathfrak{g}/\tilde{\mathfrak{k}})$:

Let us come back to the setting of Proposition 3.2. Recall that σ_j^+ is the minimal K -type of I_j . Denote the highest weight of the $G_{\mathbb{C}}$ representation V_j by (λ_L, λ_R) , and define the K -type

$$(40) \quad \sigma_j := E_{\lambda_L - w_0 \lambda_R},$$

Let σ_n be the PRV component of $\sigma_j \otimes \sigma_j^+$.

Lemma 4.7. *In Proposition 3.2 (b) and (c), the K -type σ_n occurs with multiplicity one in $\wedge^{n-1}(\mathfrak{g}/\tilde{\mathfrak{k}})$.*

Proof. In case (b) of Proposition 3.2, σ_j^+ is given by (30). On the other hand, σ_j has highest weight $(-1, \dots, -1, 2k_\eta - 2n\kappa + n - 1)$. Thus the PRV component of $\sigma_j \otimes \sigma_j^+$ is $E_{(n-1, -1, \dots, -1)}$.

In case (c) of Proposition 3.2, σ_j^+ is given by (31). On the other hand, σ_j has highest weight $(2k_\eta - 2n\kappa - n + 1, 1, \dots, 1, 1)$. Thus the PRV component of $\sigma_j \otimes \sigma_j^+$ is $E_{(1, \dots, 1, 1-n)}$.

Now in each case, the desired conclusion follows from Lemma 4.3. \square

4.4. Analysis of $\tau_n \otimes \tau_n \otimes \sigma_n$. Let $\tau_n \boxtimes \tau_n \boxtimes \sigma_n$ be the outer tensor product of τ_n , τ_n and σ_n . That is, it has the same underlying space as $\tau_n \otimes \tau_n \otimes \sigma_n$, but it is a representation of $K \times K \times K$. When restricted to K via the embedding $k \mapsto (k, k, k)$, it becomes the usual tensor product $\tau_n \otimes \tau_n \otimes \sigma_n$. Note that

$$\mathfrak{g}/\tilde{\mathfrak{k}} \oplus \mathfrak{g}/\tilde{\mathfrak{k}} \oplus \mathfrak{g}/\tilde{\mathfrak{k}} \cong \tilde{\mathfrak{p}} \oplus \tilde{\mathfrak{p}} \oplus \tilde{\mathfrak{p}}$$

as $K \times K \times K$ representations. Given a vector $A \in \tilde{\mathfrak{p}}$, we denote by $A^1 = (A, 0, 0)$, $A^2 = (0, A, 0)$ and $A^3 = (0, 0, A)$. Therefore,

$$(41) \quad e_{ij}^l, 1 \leq i \neq j \leq n; e_{11}^l - e_{kk}^l, 2 \leq k \leq n,$$

where $l = 1, 2$ or 3 , is a basis of $\tilde{\mathfrak{p}} \oplus \tilde{\mathfrak{p}} \oplus \tilde{\mathfrak{p}}$ consisting of weight vectors.

Lemma 4.8. *The representation $\tau_n \boxtimes \tau_n \boxtimes \sigma_n$ occurs with multiplicity one in $\wedge^{2b_n + c_n}(\mathfrak{g}/\tilde{\mathfrak{k}} \oplus \mathfrak{g}/\tilde{\mathfrak{k}} \oplus \mathfrak{g}/\tilde{\mathfrak{k}})$.*

Proof. We consider the case $\sigma_n = E_{(n-1, -1, \dots, -1)}$. The other case $\sigma_n = E_{(1, \dots, 1, 1-n)}$ is similar. Firstly, it is direct to check that

$$(42) \quad e_{12}^1 \wedge \dots \wedge e_{n-1, n}^1 \wedge e_{12}^2 \wedge \dots \wedge e_{n-1, n}^2 \wedge (e_{11}^3 - e_{22}^3) \wedge \dots \wedge (e_{11}^3 - e_{nn}^3) \in \wedge^{2b_n+c_n}(\tilde{\mathfrak{p}} \oplus \tilde{\mathfrak{p}} \oplus \tilde{\mathfrak{p}})$$

is a highest weight vector with weight

$$(n-1, \dots, 1-n; n-1, \dots, 1-n; n-1, -1, \dots, -1).$$

Thus $\tau_n \boxtimes \tau_n \boxtimes \sigma_n$ occurs in $\wedge^{2b_n+c_n}(\tilde{\mathfrak{p}} \oplus \tilde{\mathfrak{p}} \oplus \tilde{\mathfrak{p}})$.

Now suppose that v_0 is the highest weight vector of an occurrence of $\tau_n \boxtimes \tau_n \boxtimes \sigma_n$ in $\wedge^{2b_n+c_n}(\tilde{\mathfrak{p}} \oplus \tilde{\mathfrak{p}} \oplus \tilde{\mathfrak{p}})$. Based on the proofs of Lemmas 4.5 and 4.7, it is now easy to see that v_0 is a scalar multiple of (42). Thus $\tau_n \boxtimes \tau_n \boxtimes \sigma_n$ occurs with multiplicity one in $\wedge^{2b_n+c_n}(\tilde{\mathfrak{p}} \oplus \tilde{\mathfrak{p}} \oplus \tilde{\mathfrak{p}})$. \square

For simplicity, we put $K^3 := K \times K \times K$. Using Lemma 4.8, we fix a nonzero element

$$(43) \quad \eta_n \in \text{Hom}_{K^3}(\wedge^{2b_n+c_n}(\mathfrak{g}/\tilde{\mathfrak{k}} \oplus \mathfrak{g}/\tilde{\mathfrak{k}} \oplus \mathfrak{g}/\tilde{\mathfrak{k}}), \tau_n \boxtimes \tau_n \boxtimes \sigma_n).$$

Write

$$\iota_n : \wedge^{2b_n+c_n}(\mathfrak{g}/\tilde{\mathfrak{k}}) \rightarrow \wedge^{2b_n+c_n}(\mathfrak{g}/\tilde{\mathfrak{k}} \oplus \mathfrak{g}/\tilde{\mathfrak{k}} \oplus \mathfrak{g}/\tilde{\mathfrak{k}})$$

for the natural embedding.

Lemma 4.9. *We have $\dim(\tau_n \otimes \tau_n \otimes \sigma_n)^K = 1$, where $\sigma_n = E_{(n-1, -1, \dots, -1)}$ or $E_{(1, \dots, 1, 1-n)}$.*

This lemma follows directly from Proposition 3.1, we omit the details.

Lemma 4.10. *The composition*

$$(44) \quad \wedge^{2b_n+c_n}(\mathfrak{g}/\tilde{\mathfrak{k}}) \xrightarrow{\iota_n} \wedge^{2b_n+c_n}(\mathfrak{g}/\tilde{\mathfrak{k}} \oplus \mathfrak{g}/\tilde{\mathfrak{k}} \oplus \mathfrak{g}/\tilde{\mathfrak{k}}) \xrightarrow{\eta_n} \tau_n \boxtimes \tau_n \boxtimes \sigma_n$$

is nonzero. Its image is equal to $(\tau_n \otimes \tau_n \otimes \sigma_n)^K$.

Proof. We consider the case $\sigma_n = E_{(n-1, -1, \dots, -1)}$. The other case $\sigma_n = E_{(1, \dots, 1, 1-n)}$ is similar. Let us equip $\tilde{\mathfrak{p}}$ with a K -invariant positive definite Hermitian form $\langle \cdot, \cdot \rangle$ such that (34) is an orthogonal basis. It induces a K^3 -invariant positive definite Hermitian form $\langle \cdot, \cdot \rangle_\wedge$ on $\wedge^{2b_n+c_n}(\tilde{\mathfrak{p}} \oplus \tilde{\mathfrak{p}} \oplus \tilde{\mathfrak{p}})$. Note that

$$\wedge^{2b_n+c_n}(\mathfrak{g}/\tilde{\mathfrak{k}} \oplus \mathfrak{g}/\tilde{\mathfrak{k}} \oplus \mathfrak{g}/\tilde{\mathfrak{k}}) \xrightarrow{\eta_n} \tau_n \boxtimes \tau_n \boxtimes \sigma_n$$

is a scalar multiple of the orthogonal projection. By Remarks 4.4 and 4.6, in the unique realization of $\tau_n \boxtimes \tau_n \boxtimes \sigma_n$ in $\wedge^{2b_n+c_n}(\tilde{\mathfrak{p}} \oplus \tilde{\mathfrak{p}} \oplus \tilde{\mathfrak{p}})$ one can find a vector of the form

$$w_0 := v_0 \wedge v'_0 \wedge ((e_{11}^3 - e_{22}^3) \wedge \dots \wedge (e_{11}^3 - e_{nn}^3) + \text{other terms}),$$

where $v_0 = e_{12}^1 \wedge \dots \wedge e_{n-1, n}^1$, $v'_0 = e_{21}^2 \wedge \dots \wedge e_{n, n-1}^2$. Here each linear component in “other terms” is different from $(e_{11}^3 - e_{22}^3) \wedge \dots \wedge (e_{11}^3 - e_{nn}^3)$, and has the form $A_1^3 \wedge \dots \wedge A_{n-1}^3$, where each A_i is in (34), see (36). Put

$$u_0 := e_{12} \wedge \dots \wedge e_{n-1, n} \wedge e_{21} \wedge \dots \wedge e_{n, n-1} \wedge (e_{11} - e_{22}) \wedge \dots \wedge (e_{11} - e_{nn}) \in \wedge^{2b_n+c_n} \tilde{\mathfrak{p}}.$$

Then one sees easily that the “other terms” has no contribution in $\langle \iota_n(u_0), w_0 \rangle_\wedge$, and that the latter is nonzero. Thus $\eta_n \circ \iota_n$ is nonzero.

The image of the composition (44) is a nonzero subspace of $(\tau_n \otimes \tau_n \otimes \sigma_n)^K$. By Lemma 4.9, the latter space is one dimensional. Therefore the second assertion follows. \square

4.5. **Analysis of F_ξ^\vee .** Recall from the introduction that

$$F_\xi^\vee := F_\mu^\vee \otimes F_\nu^\vee \otimes V_j.$$

As a $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ representation, $F_\mu^\vee \cong E_{\mu_L}^\vee \boxtimes E_{\mu_R}^\vee$. When restricted to K , we have

$$F_\mu^\vee \cong E_{\mu_L}^\vee \otimes E_{\mu_R}.$$

Similarly, $F_\nu^\vee \cong E_{\nu_L}^\vee \otimes E_{\nu_R}$, and $V_j \cong E_{\lambda_L} \otimes E_{\lambda_R}^\vee$ as representations of K . Recall the K -types $\tau_\mu = E_{\mu_L - w_0 \mu_R}$, $\tau_\nu = E_{\nu_L - w_0 \nu_R}$ and $\sigma_j = E_{\lambda_L - w_0 \lambda_R}$. Put

$$(45) \quad \tau_\xi^\vee := \tau_\mu^\vee \otimes \tau_\nu^\vee \otimes \sigma_j,$$

and

$$(46) \quad \tau_\xi^+ := \tau_\mu^+ \otimes \tau_\nu^+ \otimes \sigma_j^+.$$

Lemma 4.11. *We have $\dim(\tau_\xi^\vee)^K = 1$.*

Proof. We only prove it for Proposition 3.2 (b), where $\sigma_j = \det^{-1} \otimes \text{Sym}^{2k_\eta - 2n\kappa + n}$. Thus

$$\text{Hom}_K(\mathbb{C}, \tau_\xi^\vee) \cong \text{Hom}_K(\tau_\mu \otimes \tau_\nu \otimes \text{Sym}^{2n\kappa - 2k_\eta - n}, \det^{-1}).$$

By Proposition 3.1, the latter space is one dimensional if and only if $2M_{\mu,\nu}^{n+1} - 2\kappa \leq -1 \leq 2m_{\mu,\nu}^n - 2\kappa$. This is exactly (27). \square

Lemma 4.12. *We have $\dim(\tau_\xi^+)^K = 1$.*

Proof. We only prove it for case (b) of Proposition 3.2. Indeed, by Proposition 3.1, $\text{Hom}_K(\mathbb{C}, \tau_\xi^+)$ is one dimensional if and only if $2M_{\mu,\nu}^{n+1} - 2\kappa \leq 0 \leq 2m_{\mu,\nu}^n - 2\kappa + 2$. On one hand, by (27), we have

$$2M_{\mu,\nu}^{n+1} - 2\kappa \leq 2(\kappa - \frac{1}{2}) - 2\kappa \leq -1 < 0.$$

On the other hand, by (25), we have

$$2m_{\mu,\nu}^n - 2\kappa + 2 \geq 2(\kappa - \frac{1}{2}) - 2\kappa + 2 \geq 1 > 0.$$

This finishes the proof. \square

Lemma 4.13. *The $K \times K \times K$ representation τ_ξ^\vee occurs with multiplicity one in F_ξ^\vee . Moreover, every non-zero element of $\text{Hom}_{G_\mathbb{C}}(F_\xi^\vee, \mathbb{C})$ does not vanish on $\tau_\xi^\vee \subset F_\xi^\vee$.*

Proof. Note that τ_μ^\vee is the Cartan component of F_μ^\vee . Similarly, τ_ν^\vee (resp. σ_j) is the Cartan component of F_ν^\vee (resp. V_j). Thus the $K \times K \times K$ representation τ_ξ^\vee occurs with multiplicity one in F_ξ^\vee . This proves the first assertion.

We fix a nonzero element ϕ_F in $\text{Hom}_{G_\mathbb{C}}(F_\xi^\vee, \mathbb{C})$. In view of the proof of Proposition 3.3, we have $\phi_F = \phi_L \otimes \phi_R$, where ϕ_L is a nonzero element in $\text{Hom}_K(E_{\mu_L}^\vee \otimes E_{\nu_L}^\vee \otimes E_{\lambda_L}, \mathbb{C})$, and ϕ_R is a nonzero element in $\text{Hom}_K(E_{\mu_R} \otimes E_{\nu_R} \otimes E_{\lambda_R}^\vee, \mathbb{C})$.

We denote a nonzero highest (resp. lowest) weight vector of E_{λ_L} by λ_L^+ (resp. λ_L^-). Note that as a K -type, $E_{\lambda_L}^\vee$ can be realized on the same space as E_{λ_L} , and then λ_L^+ (resp. λ_L^-) becomes a lowest (resp. highest) weight vector. We interpret μ_L^+ , μ_L^- , ν_L^+ and ν_L^- similarly.

Denote all the lower triangular matrices in $\mathfrak{gl}_n := \mathfrak{gl}(n, \mathbb{C})$ with real diagonal entries by \mathfrak{b}_1 , and denote those upper triangular matrices with purely imaginary diagonal entries by \mathfrak{b}_2 . We consider the subalgebra $\mathfrak{b} := \mathfrak{b}_1 \oplus \mathfrak{b}_2 \oplus \mathfrak{gl}_n$ of $\mathfrak{gl}_n \oplus \mathfrak{gl}_n \oplus \mathfrak{gl}_n$. Let \mathfrak{h} be the diagonal embedding of \mathfrak{gl}_n in $\mathfrak{gl}_n \oplus \mathfrak{gl}_n \oplus \mathfrak{gl}_n$. Then it is direct to check that $\mathfrak{h} \cap \mathfrak{b} = \{0\}$. Then by dimension counting, we have $\mathfrak{gl}_n \oplus \mathfrak{gl}_n \oplus \mathfrak{gl}_n = \mathfrak{h} \oplus \mathfrak{b}$. Therefore,

$$E_{\mu_L}^\vee \otimes E_{\nu_L}^\vee \otimes E_{\lambda_L} = U(\mathfrak{h})U(\mathfrak{b})(\mu_L^+ \otimes \nu_L^- \otimes \lambda_L^+) = U(\mathfrak{h})(\mu_L^+ \otimes \nu_L^- \otimes E_{\lambda_L}).$$

Thus ϕ_L does not vanish on the space $\mu_L^+ \otimes \nu_L^- \otimes E_{\lambda_L}$. Similarly, ϕ_R does not vanish on the space $\mu_R^- \otimes \nu_R^+ \otimes E_{\lambda_R}^\vee$.

Denote by $(E_{\lambda_L})_\lambda$ the λ weight space of E_{λ_L} . Now let us adopt the setting of Proposition 3.2 (c), and prove the second assertion. The case of Proposition 3.2 (b) is similar. Note that each weight space of E_{λ_L} and $E_{\lambda_R}^\vee$ is one dimensional. We choose a basis of E_{λ_L} (resp. $E_{\lambda_R}^\vee$) consisting of weight vectors. Since ϕ_L maps $E_{\mu_L}^\vee \otimes E_{\nu_L}^\vee \otimes E_{\lambda_L}$ onto the trivial representation K -equivariantly, we have that ϕ_L does not vanish on $\mu_L^+ \otimes \nu_L^- \otimes (E_{\lambda_L})_\lambda$ if and only if the latter space has weight zero. That is, if and only if $\lambda = \mu_L + w_0\nu_L$. We put $\lambda_L^0 := \mu_L + w_0\nu_L$, and put $a_k := \mu_k + \nu_{n+1-k} + j - 1$ for $2 \leq k \leq n$. Note that each a_k is nonnegative since j is assumed to be a critical place for $\pi_\mu \times \pi_\nu$, see (26). Now we have

$$\lambda_L - \lambda_L^0 = \sum_{k=2}^n a_k(\epsilon_1 - \epsilon_k),$$

and there exists an element $E_L \in U(\mathfrak{gl}_n)_{-\lambda_L + \lambda_L^0}$ such that

$$(47) \quad E_L \cdot \lambda_L^+ = w_L,$$

where w_L is a nonzero vector in $(E_{\lambda_L})_{\lambda_L^0}$, and it is among the chosen basis of E_{λ_L} . Here “.” means the action of \mathfrak{gl}_n on E_{λ_L} .

Similarly, ϕ_R does not vanish on $\mu_R^- \otimes \nu_R^+ \otimes (E_{\lambda_R}^\vee)_\lambda$ if and only if $\lambda = -w_0\mu_R - \nu_R$. We put $\lambda_R^0 := -w_0\mu_R - \nu_R$, and put $b_k := -j - 2\kappa + \mu_k + \nu_{n+1-k}$ for $2 \leq k \leq n$, which are all nonnegative by (26). Note that

$$-w_0\lambda_R - \lambda_R^0 = \sum_{k=2}^n b_k(\epsilon_1 - \epsilon_k),$$

and there exists an element $E_R \in U(\mathfrak{gl}_n)_{w_0\lambda_R + \lambda_R^0}$ such that

$$(48) \quad E_R \cdot \lambda_R^- = w_R,$$

where w_R is a nonzero vector in $(E_{\lambda_R}^\vee)_{\lambda_R^0}$, and it is among the chosen basis of $E_{\lambda_R}^\vee$.

The vector $\lambda_L^+ \otimes \lambda_R^-$ is a highest weight vector of the Cartan component of $E_{\lambda_L} \otimes E_{\lambda_R}^\vee$. Moreover, by (47) and (48), we have

$$(49) \quad w_0 := E_R E_L \cdot (\lambda_L^+ \otimes \lambda_R^-) = w_L \otimes w_R + \text{other terms}.$$

Here each linear component in “other terms” is different from $w_L \otimes w_R$, and has the form $w_1 \otimes w_2$, where w_1 (resp. w_2) is among the chosen basis of E_{λ_L} (resp. $E_{\lambda_R}^\vee$), and $w_1 \otimes w_2 \in$

$(E_{\lambda_L} \otimes E_{\lambda_R}^\vee)_{\lambda_L^0 + \lambda_R^0}$. We necessarily have $\phi_L(\mu_L^+ \otimes \nu_L^- \otimes w_1) = 0$ or $\phi_R(\mu_R^- \otimes \nu_R^+ \otimes w_2) = 0$. Therefore,

$$\phi_F((\mu_L^+ \boxtimes \mu_R^-) \otimes (\nu_L^- \boxtimes \nu_R^+) \otimes (w_1 \boxtimes w_2)) = \phi_L(\mu_L^+ \otimes \nu_L^- \otimes w_1) \phi_R(\mu_R^- \otimes \nu_R^+ \otimes w_2) = 0.$$

Thus by (49),

$$\phi_F((\mu_L^+ \boxtimes \mu_R^-) \otimes (\nu_L^- \boxtimes \nu_R^+) \otimes w_0) = \phi_L(\mu_L^+ \otimes \nu_L^- \otimes w_L) \phi_R(\mu_R^- \otimes \nu_R^+ \otimes w_R) \neq 0.$$

Note that $\mu_L^+ \otimes \mu_R^-$ is a lowest weight vector of τ_μ^\vee , and that $\nu_L^- \otimes \nu_R^+$ is a highest weight vector of τ_ν^\vee . Moreover, the vector w_0 defined in (49) lives in σ_j , the Cartan component of $E_{\lambda_L} \otimes E_{\lambda_R}^\vee$. We conclude that ϕ_F does not vanish on $\tau_\xi^\vee = \tau_\mu^\vee \otimes \tau_\nu^\vee \otimes \sigma_j$. This finishes the proof. \square

4.6. Analysis of linear functionals on PRV components. The following lemma is taken from Section 2.1 of [21].

Lemma 4.14. *Let α_1 and α_2 be two irreducible representations of a compact connected Lie group L . Let α_3 be the Cartan component of $\alpha_1 \otimes \alpha_2$. Let $f : \alpha_1 \otimes \alpha_2 \rightarrow \alpha_3$ be a nonzero L -equivariant linear map. Then f maps all nonzero decomposable vectors (namely, vectors of the form $u \otimes v \in \alpha_1 \otimes \alpha_2$) to nonzero vectors.*

Let σ_1 and σ_2 be two irreducible representations of K^3 , and write σ_3 for their Cartan component. The remaining part of this section is devoted to deducing an analog of Proposition 2.16 of [16]. Since our situation is slightly more complicated, we give a proof here.

Proposition 4.15. *Assume that $\dim(\sigma_i)^K = 1$ for $1 \leq i \leq 3$. Let ϕ_1 (resp. ϕ_3) be any nonzero K -invariant linear functional on σ_1^\vee (resp. σ_3). Then $\phi_1 \otimes \phi_3$ does not vanish on the PRV component*

$$\sigma_2 \subset \sigma_1^\vee \otimes \sigma_3.$$

Proof. Let $\sigma_i = \alpha_i \otimes \beta_i \otimes \gamma_i$ for $1 \leq i \leq 3$. Fix a generator $v_2 \in (\alpha_2 \otimes \beta_2 \otimes \gamma_2)^K$. It is routine to check that the following diagram commutes:

$$\begin{array}{ccc} \text{Hom}_{K^3}(\alpha_2 \otimes \beta_2 \otimes \gamma_2, \alpha_1^\vee \otimes \beta_1^\vee \otimes \gamma_1^\vee \otimes \alpha_3 \otimes \beta_3 \otimes \gamma_3) & \xrightarrow{f \mapsto ((\phi_1 \otimes \phi_3) \circ f)(v_2)} & \mathbb{C} \\ \downarrow \cong & & \downarrow = \\ \text{Hom}_K(\gamma_3^\vee, \gamma_1^\vee \otimes \gamma_2^\vee) \otimes \text{Hom}_K(\beta_3^\vee, \beta_1^\vee \otimes \beta_2^\vee) \otimes \text{Hom}_K(\alpha_1 \otimes \alpha_2, \alpha_3) & \xrightarrow{f \otimes g \otimes h \mapsto \phi_3(h \circ (\phi_1 \otimes v_2) \circ (f \otimes g))} & \mathbb{C} \end{array}$$

where the left vertical arrow is the canonical isomorphism, and in the bottom horizontal arrow, we view

$$\begin{aligned} v_2 \in \text{Hom}_K(\beta_2^\vee \otimes \gamma_2^\vee, \alpha_2) &= (\alpha_2 \otimes \beta_2 \otimes \gamma_2)^K, \\ \phi_1 \in \text{Hom}_K(\beta_1^\vee \otimes \gamma_1^\vee, \alpha_1) &= \text{Hom}_K(\alpha_1^\vee \otimes \beta_1^\vee \otimes \gamma_1^\vee, \mathbb{C}), \text{ and} \\ h \circ (\phi_1 \otimes v_2) \circ (f \otimes g) &\in \text{Hom}_K(\gamma_3^\vee \otimes \beta_3^\vee, \alpha_3) = (\alpha_3 \otimes \beta_3 \otimes \gamma_3)^K. \end{aligned}$$

Now it suffices to check that the bottom horizontal arrow of the diagram is nonzero. We pick up a generator

$$f_0 \otimes g_0 \otimes h_0 \in \text{Hom}_K(\gamma_3^\vee, \gamma_1^\vee \otimes \gamma_2^\vee) \otimes \text{Hom}_K(\beta_3^\vee, \beta_1^\vee \otimes \beta_2^\vee) \otimes \text{Hom}_K(\alpha_1 \otimes \alpha_2, \alpha_3).$$

Note that β_3^\vee (resp. γ_3^\vee) is the Cartan component of $\beta_1^\vee \otimes \beta_2^\vee$ (resp. $\gamma_1^\vee \otimes \gamma_2^\vee$). Let u_3 be a nonzero highest weight vector of β_3^\vee , and let v_3 be a nonzero lowest weight vector of γ_3^\vee . By Lemma 2.11 of [16], $g_0(u_3)$ (resp. $f_0(v_3)$) is a nonzero decomposable vector in $\beta_1^\vee \otimes \beta_2^\vee$ (resp. $\gamma_1^\vee \otimes \gamma_2^\vee$). Therefore, $(\phi_1 \otimes v_2)(f_0 \otimes g_0)(v_3 \otimes u_3)$ is a decomposable vector in $\alpha_1 \otimes \alpha_2$. We will soon show that it is actually nonzero. Now Lemma 4.14 implies that

$$(h_0 \circ (\phi_1 \otimes v_2) \circ (f_0 \otimes g_0))(v_3 \otimes u_3) \neq 0.$$

Thus $h_0 \circ (\phi_1 \otimes v_2) \circ (f_0 \otimes g_0)$ is a nonzero generator of the one dimensional space

$$\text{Hom}_K(\gamma_3^\vee \otimes \beta_3^\vee, \alpha_3) = (\alpha_3 \otimes \beta_3 \otimes \gamma_3)^K.$$

Since ϕ_3 does not vanish on $(\alpha_3 \otimes \beta_3 \otimes \gamma_3)^K$, this shows that the bottom horizontal arrow is nonzero, as desired.

It remains to show that $(\phi_1 \otimes v_2)(f_0 \otimes g_0)(v_3 \otimes u_3)$ is nonzero. For $i = 1, 2$, let μ_i be the highest weight of β_i , and let ν_i be the highest weight of γ_i ; denote by μ_i^+ (resp. μ_i^-) a nonzero highest (resp. lowest) weight vector of β_i ; denote by ν_i^+ (resp. ν_i^-) a nonzero highest (resp. lowest) weight vector of γ_i . We realize β_1^\vee on the same space of β_1 . Then, say, μ_1^+ is a lowest weight vector of β_1^\vee . Up to nonzero scalars,

$$g_0(u_3) = \mu_1^- \otimes \mu_2^-, \quad f_0(v_3) = \nu_1^+ \otimes \nu_2^+.$$

Using the same technique as in Lemma 4.13, one has that $\phi_1(\mu_1^- \otimes \nu_1^+ \otimes \alpha_1^\vee) \neq 0$. Moreover, then one deduces that $\phi_1(\mu_1^- \otimes \nu_1^+ \otimes (\alpha_1^\vee)_\lambda)$ is nonzero if and only if $\lambda = w_0\mu_1 + \nu_1$. Here $(\alpha_1^\vee)_\lambda$ denotes the λ weight space of α_1^\vee . Therefore, when ϕ_1 is viewed as a K -equivariant homomorphism from $\beta_1^\vee \otimes \gamma_1^\vee$ to α_1 , we have that

$$\phi_1(\mu_1^- \otimes \nu_1^+) \in (\alpha_1)_{-w_0\mu_1 - \nu_1}$$

is nonzero. Similarly,

$$v_2(\mu_2^- \otimes \nu_2^+) \in (\alpha_2)_{-w_0\mu_2 - \nu_2}$$

is nonzero. Thus

$$(\phi_1 \otimes v_2)(f_0 \otimes g_0)(v_3 \otimes u_3) = \phi_1(\mu_1^- \otimes \nu_1^+) \otimes v_2(\mu_2^- \otimes \nu_2^+)$$

is nonzero. This finishes the proof. \square

5. INFINITE DIMENSIONAL REPRESENTATIONS

Recall the pure weights μ and ν from (15) and (18), respectively. Assume that μ and ν are compatible. Furthermore, we adopt the assumption (12) from now on. That is, we assume that κ is an half integer and fix $j = -\kappa + \frac{1}{2}$, which is a critical place for $\pi_\mu \times \pi_\nu$. Recall the K -types τ_μ^+ , τ_ν^+ and σ_j^+ from the previous section. Recall from (46) that $\tau_\xi^+ := \tau_\mu^+ \otimes \tau_\nu^+ \otimes \sigma_j^+$. By Lemmas 4.1 and 4.2, the K^3 -representation τ_ξ^+ occurs with multiplicity one in π_ξ .

Recall that $G = \text{GL}(n, \mathbb{K})$. Fix $H = \text{SL}(n, \mathbb{K})$. This section aims to prove the following proposition.

Proposition 5.1. *Under the assumption (12), every nonzero element of $\text{Hom}_G(\pi_\xi, \mathbb{C})$ does not vanish on $\tau_\xi^+ \subset \pi_\xi$.*

By Section 1, the representations $\pi_\mu|_H$ and $\pi_\nu|_H$ are unitarizable and tempered. Also, $I_j|_H$ is unitarizable under the assumption (12). We fix a H^3 -invariant positive definite continuous Hermitian form $\langle \cdot, \cdot \rangle_\xi := \langle \cdot, \cdot \rangle_{\pi_\mu} \otimes \langle \cdot, \cdot \rangle_{\pi_\nu} \otimes \langle \cdot, \cdot \rangle_{I_j}$ on π_ξ .

Lemma 5.2. *The integrals in*

$$(50) \quad \begin{aligned} \pi_\xi \times \pi_\xi &\rightarrow \mathbb{C}, \\ (u, v) &\mapsto \int_H \langle h.u, v \rangle_\xi dh \end{aligned}$$

converge absolutely for all $u, v \in \pi_\xi$, and yields a continuous H -invariant Hermitian form on π_ξ . Here “ dh ” denotes a Haar measure on H .

Proof. We may and do assume that

$$u = u_1 \otimes u_2 \otimes u_3 \quad \text{and} \quad v = v_1 \otimes v_2 \otimes v_3,$$

where $u_1, v_1 \in \pi_\mu$, $u_2, v_2 \in \pi_\nu$ and $u_3, v_3 \in I_j$. Let $H = K_0 A^+ K_0$ be the Cartan decomposition of H , where

$$A^+ := \{\text{diag}(a_1, a_2, \dots, a_n) \mid \prod_{i=1}^n a_i = 1, \text{ and } a_i \geq a_{i+1} > 0 \text{ for all } 1 \leq i \leq n-1\}$$

is the positive Weyl chamber, and $K_0 = \text{SU}(n)$ is the maximal compact subgroup of H . It is well known that $\int_H |\langle h.u, v \rangle_\xi| dh < \infty$ if and only if $\int_{A^+} |\langle a.u, v \rangle_\xi| \delta_B(a) d^\times a < \infty$, where $d^\times a$ is a Haar measure on A^+ , and δ_B is the modular character of the standard Borel subgroup B of H .

Denote by Ξ_{K_0} the Harish-Chandra function on H associated to the maximal compact subgroup K_0 (see [18, Section 4.5.3]). By [15, Theorem 1.2], there is a continuous seminorm $|\cdot|_{\pi_\mu}$ on π_μ such that

$$(51) \quad |\langle \pi_\mu(a).u_1, v_1 \rangle_{\pi_\mu}| \leq \Xi_{K_0}(a) \cdot |u_1|_{\pi_\mu} |v_1|_{\pi_\mu}, \quad u_1, v_1 \in \pi_\mu, a \in A^+.$$

Analogously, there is a continuous seminorm $|\cdot|_{\pi_\nu}$ on π_ν such that

$$(52) \quad |\langle \pi_\nu(a).u_2, v_2 \rangle_{\pi_\nu}| \leq \Xi_{K_0}(a) \cdot |u_2|_{\pi_\nu} |v_2|_{\pi_\nu}, \quad u_2, v_2 \in \pi_\nu, a \in A^+.$$

Applying Langlands classification (due to Zhelobenko in the case of complex groups [22]), one sees that I_j is isomorphic to the irreducible quotient of $\text{Ind}_B^H(\chi \otimes 1)$, where $B = LN$ is the Borel subgroup with Levi factor L and unipotent radical N , and

$$\chi(a) = (a_1 \overline{a_1})^{\frac{n-2}{2}-\kappa} (a_2 \overline{a_2})^{\frac{n-4}{2}-\kappa} \cdots (a_{n-1} \overline{a_{n-1}})^{-\frac{n-2}{2}-\kappa} a_n^{(n-1)\kappa-k_\eta} \overline{a_n}^{k_\eta-(n+1)\kappa}$$

for $a = \text{diag}(a_1, a_2, \dots, a_n) \in L$. By [18, Lemma 5.2.8], there are constants $c, d > 0$ such that

$$(53) \quad |\langle I_j(a).u_3, v_3 \rangle_{I_j}| \leq c \cdot \chi(a) \Xi_{K_0}(a) (1 + \log \|a\|)^d, \quad u_3, v_3 \in I_j, a \in A^+,$$

where $\|\cdot\|$ is certain norm on H . By the estimate of Ξ_{K_0} in [18, Theorem 4.5.3] and (51), (52), (53), the integral $\int_{A^+} |\langle a.u, v \rangle_\xi| \delta_B(a) d^\times a$ is convergent. Thus $\int_H \langle h.u, v \rangle_\xi dh$ converges absolutely. The map defined by (50) yields an H -invariant Hermitian form since H is unimodular. \square

Write $H = SK_0$ for the Cartan decomposition of H , where

$$S := \{\text{positive definite Hermitian matrices with determinant } 1\}.$$

Lemma 5.3. ([14, Theorem 1.4]) *For every nonzero vector u in the minimal K_0^3 -type τ_ξ^+ of π_ξ , one has that*

$$\langle g.u, u \rangle_\xi > 0, \quad \text{for all } g \in S^3.$$

By Lemma 4.12, the space $(\tau_\xi^+)^{K_0} \neq 0$. Take a nonzero element $v_\xi \in (\tau_\xi^+)^{K_0}$. Then we have the following positivity lemma.

Lemma 5.4. *The Hermitian form defined in (50) is positive definite on the one dimensional space $\mathbb{C}v_\xi$.*

Proof. Denote by dk the Haar measure on K_0 such that the volume of K_0 is 1, and let ds be a certain positive measure on S . By Lemma 5.3, one has that

$$\begin{aligned} & \int_H \langle h.v_\xi, v_\xi \rangle_\xi dh \\ &= \int_S \int_{K_0} \langle sk.v_\xi, v_\xi \rangle_\xi dk ds \\ &= \int_S \langle s.v_\xi, v_\xi \rangle_\xi ds > 0. \end{aligned}$$

□

Lemma 5.5. *Under the assumption (12), there exists an element of $\text{Hom}_G(\pi_\xi, \mathbb{C})$ which does not vanish on $\tau_\xi^+ \subset \pi_\xi$.*

Proof. Define

$$\begin{aligned} \phi : \pi_\xi &\rightarrow \mathbb{C}, \\ u &\mapsto \int_H \langle h.u, v_\xi \rangle_\xi dh. \end{aligned}$$

Then $\phi \in \text{Hom}_H(\pi_\xi, \mathbb{C})$. By Lemma 5.4, ϕ does not vanish on τ_ξ^+ . Since the representation π_ξ has a trivial central character when restricted to G , the map ϕ is G -intertwining, which completes the proof of the lemma. □

Lemma 5.6. *Suppose that μ and ν are compatible (Definition 3.6), and j is a critical place for $\pi_\mu \times \pi_\nu$. Then the inequality*

$$(54) \quad \dim \text{Hom}_G(\pi_\xi, \mathbb{C}) \leq 1$$

holds.

Proof. Denote by $\mathcal{S}(\mathbb{C}^n)$ the space of Schwartz functions on \mathbb{C}^n . The space $\mathcal{S}(\mathbb{C}^n)$ carries an action of G defined by

$$g.f(v) := f(vg), \quad v \in \mathbb{C}^n, f \in \mathcal{S}(\mathbb{C}^n), g \in G.$$

Let $Z \cong \mathbb{C}^\times$ be the center of G , and let P_0 be the subgroup of P with the last row being $v_0 := (0, \dots, 0, 1)$. Then $P = P_0 \cdot Z$. Define a character χ_j of Z as

$$\chi_j(z) := z^{-nj-k_\eta} \cdot \bar{z}^{-nj+k_\eta-2n\kappa}, \quad z \in Z.$$

Then

$$I_j = {}^u\text{Ind}_P^G H_j = |\det|_{\mathbb{C}}^j \otimes {}^u\text{Ind}_P^G (1 \otimes \chi_j),$$

where ${}^u\text{Ind}$ stands for the non-normalized smooth induction.

Write $\mathcal{S}(\mathbb{C}^n \setminus \{0\})$ for the space of Schwartz functions on $\mathbb{C}^n \setminus \{0\}$. The linear map

$$\begin{aligned} \mathcal{S}(\mathbb{C}^n \setminus \{0\}) &\rightarrow C^\infty(G), \\ \varphi &\mapsto (g \mapsto \int_Z \varphi(v_0 g z) \chi_j(z)^{-1} dh) \end{aligned}$$

induces a G -intertwining isomorphism

$$(55) \quad \mathcal{S}(\mathbb{C}^n \setminus \{0\})_{Z, \chi_j} \cong \text{Ind}_P^G(1 \otimes \chi_j),$$

where $(\cdot)_{Z, \chi_j}$ indicates the maximal Hausdorff quotient space on which Z acts through the character χ_j . Suppose we are in case (b) of Proposition 3.2, then

$$-nj - k_\eta \geq 0 \quad \text{and} \quad -nj + k_\eta - 2n\kappa \leq -n.$$

By Theorem 1.1 (a) of [20], the natural imbedding $\mathcal{S}(\mathbb{C}^n \setminus \{0\}) \hookrightarrow \mathcal{S}(\mathbb{C}^n)$ induces an isomorphism

$$(56) \quad \mathcal{S}(\mathbb{C}^n \setminus \{0\})_{Z, \chi_j} \cong \mathcal{S}(\mathbb{C}^n)_{Z, \chi_j}$$

of representations of G . Hence

$$(57) \quad I_j \cong |\det|_{\mathbb{C}}^j \otimes \mathcal{S}(\mathbb{C}^n \setminus \{0\})_{Z, \chi_j} \cong |\det|_{\mathbb{C}}^j \otimes \mathcal{S}(\mathbb{C}^n)_{Z, \chi_j}.$$

By [17, Theorem C], for all irreducible Casselman-Wallach representations π_1, π_2 and every character χ of G , one has that

$$(58) \quad \dim \text{Hom}_G(\pi_1 \otimes \pi_2 \otimes \mathcal{S}(\mathbb{C}^n), \chi) \leq 1.$$

Now the inequality (54) is a straightforward consequence of (57) and (58). Analogous analysis implies that (54) also holds for case (c) of Proposition 3.2. \square

By Lemmas 5.5 and 5.6, it is easy to see that Proposition 5.1 holds.

6. THE RELATIVE COHOMOLOGY SPACES

Note that the center \mathbb{K}^\times of $GL(n, \mathbb{K})$ acts trivially on $\pi_\mu \otimes F_\mu^\vee$. Recall that $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$, and that $\tilde{\mathfrak{k}}$ is the complexified Lie algebra of $\tilde{K} = GU(n)$. Now we have

$$\begin{aligned} H^{b_n}(\mathfrak{g}, GU(n); \pi_\mu \otimes F_\mu^\vee) &= H^{b_n}(\mathfrak{sl}_n(\mathbb{C}) \times \mathfrak{sl}_n(\mathbb{C}), SU(n); \pi_\mu \otimes F_\mu^\vee) \\ &= \text{Hom}_{SU(n)}(\wedge^{b_n}(\mathfrak{sl}_n(\mathbb{C}) \times \mathfrak{sl}_n(\mathbb{C}) / \mathfrak{sl}_n(\mathbb{C})); \pi_\mu \otimes F_\mu^\vee) \\ &= \text{Hom}_{SU(n)}(\wedge^{b_n}(\mathfrak{g} / \tilde{\mathfrak{k}}); \pi_\mu \otimes F_\mu^\vee), \end{aligned}$$

where at the penultimate step we use Proposition 9.4.3 of [18]. Note that $\pi_\mu|_{SL_n(\mathbb{K})}$ is unitary (actually tempered). The analog also holds for $\pi_\nu \otimes F_\nu^\vee$. Now assume (12). Then $j = -\kappa + \frac{1}{2}$, note that $I_j|_{SL_n(\mathbb{K})}$ is unitary and $I_j \otimes V_j$ has a trivial central character as well. Thus, we have

$$H^{c_n}(\mathfrak{g}, GU(n); I_j \otimes V_j) = \text{Hom}_{SU(n)}(\wedge^{c_n}(\mathfrak{g} / \tilde{\mathfrak{k}}); I_j \otimes V_j).$$

7. PROOF OF THEOREM A

Recall that $G = \mathrm{GL}(n, \mathbb{K})$ and $H = \mathrm{SL}(n, \mathbb{K})$. Let us continue to assume (12). Now we are ready to prove Theorem A. We only work with Proposition 3.2 (b). Indeed, by the discussion in the previous section, we have that

$$(59) \quad H^{2b_n+c_n}(\mathfrak{g}^3, \tilde{K}^3; \pi_\xi \otimes F_\xi^\vee) = \mathrm{Hom}_{SU(n)}\left(\wedge^{2b_n+c_n}((\mathfrak{g}/\tilde{\mathfrak{k}})^3); \pi_\xi \otimes F_\xi^\vee\right).$$

Here we write $\mathfrak{g}/\tilde{\mathfrak{k}} \oplus \mathfrak{g}/\tilde{\mathfrak{k}} \oplus \mathfrak{g}/\tilde{\mathfrak{k}}$ as $(\mathfrak{g}/\tilde{\mathfrak{k}})^3$ for short. Likewise,

$$(60) \quad H^{2b_n+c_n}(\mathfrak{g}, \tilde{K}; \mathbb{C}) = \mathrm{Hom}_{SU(n)}(\wedge^{2b_n+c_n}(\mathfrak{g}/\tilde{\mathfrak{k}}); \mathbb{C}).$$

Note that $\tau_n \otimes \tau_n \otimes \sigma_n$ is the PRV-component of $\tau_\xi^+ \otimes \tau_\xi^\vee$. Write

$$\varphi_\xi : \tau_n \otimes \tau_n \otimes \sigma_n \rightarrow \pi_\xi \otimes F_\xi^\vee$$

for the inclusion

$$\tau_n \otimes \tau_n \otimes \sigma_n \subset \tau_\xi^+ \otimes \tau_\xi^\vee \subset \pi_\xi \otimes F_\xi^\vee.$$

Recall from the introduction that we can pick up a nonzero element

$$\phi_\pi \in \mathrm{Hom}_G(\pi_\xi, \mathbb{C})$$

which does not vanish on τ_ξ^+ , see Proposition 5.1. On the other hand, Lemma 4.13 guarantees the existence of a nonzero element

$$\phi_F \in \mathrm{Hom}_{G_\mathbb{C}}(F_\xi^\vee, \mathbb{C})$$

which does not vanish on τ_ξ^\vee . Recall the map $\eta_n : \wedge^{2b_n+c_n}((\mathfrak{g}/\tilde{\mathfrak{k}})^3) \rightarrow \tau_n \otimes \tau_n \otimes \sigma_n$ from (44). The composition $\varphi_\xi \circ \eta_n$ is an element of (59). Its image under the map (14) of Theorem A equals the composition map

$$(61) \quad \wedge^{2b_n+c_n}(\mathfrak{g}/\tilde{\mathfrak{k}}) \xrightarrow{\iota_n} \wedge^{2b_n+c_n}((\mathfrak{g}/\tilde{\mathfrak{k}})^3) \xrightarrow{\eta_n} \tau_n \otimes \tau_n \otimes \sigma_n \xrightarrow{\varphi_\xi} \pi_\xi \otimes F_\xi^\vee \xrightarrow{\phi_\pi \otimes \phi_F} \mathbb{C}.$$

By Proposition 4.15, the composition $(\phi_\pi \otimes \phi_F) \circ \varphi_\xi$ is nonzero. Since it is K -invariant, it does not vanish on $(\tau_n \otimes \tau_n \otimes \sigma_n)^K$. By Lemma 4.10, the latter space is equal to the image of $\eta_n \circ \iota_n$. Therefore the composition (61) is nonzero. This finishes the proof of Theorem A.

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